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Numerical Solution of Fractional Wave Equation using Crank-Nicholson Method

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Abstract: In this paper, Crank-Nicholson method for solving fractional wave equation is considered. The stability and consistency of the method are discussed by means of Greschgorin theorem and using the stability matrix analysis. Numerical solutions of some wave fractional partial differential equation models are presented. The results obtained are compared to exact solutions.

Key words: Crank-Nicholson method . fractional wave equation . stability condition . stability matrix analysis . Greschgorin theorem

INTRODUCTION

Fractional order differential equations (FDE) have been the focus of many studies due to their frequent appearance in various applications especially in the fields of fluid mechanics, viscoelasticity, biology, physics and engineering [1, 6-13] and the references cited therein). Consequently, considerable attention has been given to the solutions of fractional ordinary/partial differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques [1-4] must be used. In the following we present some basic definitions for fractional derivatives and Greschgorin theorem which are used in this paper.

Definition 1: The Riemann fractional derivative operator D_*^α of order α is defined in following form:

$$(D_*^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}dt, \quad \alpha > 0$$

where $m-1<\alpha<m, m\in N, t>0.$

Remark 1: The case a = 0 is generally called Riemann-Liouville form.

For more details on fractional derivatives definitions and its properties [5, 10, 14].

Theorem 1: (Greschgorin)

Let the matrix $A \equiv (a_{ij})$ has eigenvalues λ and define the absolute row and column sums by:

$$r_i \equiv \sum_{j=1,\, j\neq i}^{n} |a_{ij}|, \quad c_i \equiv \sum_{i=1, i\neq j}^{n} |a_{ij}|$$

Then,

(a) Each eigenvalues lies in the union of the row circles R_i, i = 1,2,…,n where

$$R_i \equiv \{ z : |z-a_{ii}| \le r_i\}$$

(b) Each eigenvalues lies in the union of the column circles C_j, j = 1,2,…,n where

$$C_j \equiv \{ z : |z-a_{jj}| \le c_j\}$$

Proof: [5].

Our aim in this paper is to study the finite difference method for solving fractional order wave equation of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + d(x,t) \tag{1}$$

on a finite domain a<x<b, 0≤t≤T. The parameter α refers to the fractional order of spatial derivatives with $1\le\alpha\le2$. The function d(x,t) is a source term and the coefficient functions c(x,t)≥0, We also assume that problem (1) is subjected to the following initial conditions:

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad \text{for } a<x<b,$$

and zero Dirichlet boundary conditions.

Clear that when α = 2, equation (1) is the classical wave equation of the following form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + d(x,t)$$

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The Grünwald-Letnikov definition is important for our purposes in this paper because it allows us to estimate $D_x^\alpha f(x)$ numerically in a simple and efficient way:

$$D_x^\alpha f(x) = \frac{1}{h^\alpha}\sum_{k=0}^{\left[\frac{x-a}{h}\right]} \omega_k f(x-kh) + O(h^p)$$

This formula is not unique because there are many different valid choices for ω_k that lead to approximations of different order p [7].

Let $\omega(z,\alpha)$ be the generating function of the coefficients ω_k, i.e,

$$\omega(z,\alpha) = \sum_{k=0}^{\infty} \omega_k z^\alpha$$

If the generating function is [15]:

$$\omega(z,\alpha) = (1-z)^\alpha$$

then we get to the backward difference formula of orderp = 1, This is also called the backward Euler formula of order 1 or, simply the Grünwald-Letnikov formula. The corresponding coefficients

$$\omega_k = (-1)^k \binom{\alpha}{k}$$

which called the normalized Grüunwald weights can be conveniently evaluated by means of the recursive formula:

$$\omega_0 = 1, \quad \omega_k = \left(1 - \frac{\alpha+1}{k}\right)\omega_{k-1} \tag{2}$$

The generating function for the backward difference formula of order p = 2 is:

$$\omega(z,\alpha) = \left(\frac{3}{2} - 2z + \frac{1}{2}z^2\right)^\alpha$$

and the generating function for the backward difference formula of order p = 3 is:

$$\omega(z,\alpha) = \left(\frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^3\right)^\alpha$$

The standard Grünwald estimates generally yield unstable finite difference equations regardless of whether the resulting finite difference method is an explicit or an implicit system [8, 9] for related discussion. Therefore, we use a shifted Grüunwald formula to estimate the α-spatial order fractional derivative [9]

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \lim_{M\to\infty} \frac{1}{h^\alpha}\sum_{k=0}^{M} \omega_k u(x-(k-1)h,t) \tag{3}$$

DISCRETIZATION FOR FRACTIONAL WAVE EQUATION

In this section the Crank-Nicloson method with shifted Grünwald formula are used to estimate the spatial α-order fractional derivative to solve numerically, the fractional wave equation of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + d(x,t) \tag{4}$$

Let us introduce the following notations:

$$t_n = n\Delta t, 0 \le t_n \le T$$

and $\Delta x = (b-a)/N = h > 0$ to be the grid size in x direction with $x_i = a+ih$ for i = 0,1,…,N. Define

$$u_i^n \approx u(x_i,t_n), c_i^n = c(x_i,t_n)$$

and

$$d_i^{n+1/2} = d(x_i,t_{n+1/2})$$

In the following two main steps are considered to build in the finite difference scheme of (4). Let U_i^n denote the numerical approximation to the exact solution u_i^n. If the shifted Grünwald estimates are substituted in the problem (4) to get Crank-Nicloson type numerical solution, the resulting finite difference equations are

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = \frac{c_i^n}{2}(\delta_{\alpha,x}u_i^{n+1} + \delta_{\alpha,x}u_i^n) + d_i^{n+1/2} \tag{5}$$

where the above fractional partial differentiation operator is defined as

$$\delta_{\alpha,x}u_i^n = \frac{1}{h^\alpha}\sum_{k=0}^{i+1}\omega_k u_{i-k+1}^n$$

Equation (5) can be written in the following form:

$$\left(1 - \frac{c_i^n(\Delta t)^2}{2}\delta_{\alpha,x}\right)u_i^{n+1} = \left(2 + \frac{c_i^n(\Delta t)^2}{2}\delta_{\alpha,x}\right)u_i^n - u_i^{n-1} + d_i^{n+1/2}(\Delta t)^2 \tag{6}$$

The difference equations (6), together with the Dirichlet boundary conditions, can be solved at time step t_{n+1} as a linear system of equations of the form:

$$(I-A)U^{n+1}=(2I+A)U^{n}-U^{n-1}+(\Delta t)^{2}D^{n+1/2} \qquad (7)$$

where

$$U^{n}=[u_0^n,u_1^n,\ldots,u_{N-1}^n,u_N^n]^T,\ D^{n+1/2}=[0,d_0^{n+1/2},\ldots,d_{N-1}^{n+1/2},0]^T$$

and I is identity matrix of dimension N-1×N-1.

To illustrate this matrix pattern, we list the corresponding first two equations for the row i = 1;2 where

$$\gamma_i=\frac{c_i^n(\Delta t)^2}{2h^{\alpha}}:$$

$$u_1^{n+1}-\gamma_1\omega_0u_2^{n+1}-\gamma_1\omega_1u_1^{n+1}-\gamma_1\omega_2u_0^{n+1}$$
$$=2u_1^n+\gamma_1\omega_0u_2^n+\gamma_1\omega_1u_1^n+\gamma_1\omega_2u_0^n-u_1^{n-1}+(\Delta t)^2d_1^{n+1/2}$$

$$u_2^{n+1}-\gamma_2\omega_0u_3^{n+1}-\gamma_2\omega_1u_2^{n+1}-\gamma_2\omega_2u_1^{n+1}-\gamma_2\omega_3u_0^{n+1}$$
$$=2u_2^n+\gamma_2\omega_0u_3^n+\gamma_2\omega_1u_2^n+\gamma_2\omega_2u_1^n+\gamma_2\omega_3u_0^n-u_2^{n-1}+(\Delta t)^2d_2^{n+1/2}$$

We refer here that, in the numerical computation we will use $U^0=u^0(x_i)$ and since

$$u_t(x,0)=\frac{u_{i,1}-u_{i,0}}{\Delta t}$$

then

$$u_{i,1}=u_{i,0}+\Delta t\,u^1(x_i)$$

hence,

$$U^1=U^0+\Delta tS$$

where

$$S=u^1(x_i)$$

Also, the matrix

$$A=[a_{ij}],i=1,2,\ldots,N-1,j=1,2,\ldots,N-1$$

is defined by

$$A_{i,j}=\begin{cases}\gamma_i\omega_{i-j+1}, & \text{for } j\le i-1\\ \gamma_i\omega_1, & \text{for } j=i\\ \gamma_i\omega_0, & \text{for } j=i+1\\ 0, & \text{for } j>i+1\end{cases}$$

Proposition 1: The fractional Crank-Nicholson discretization, using the shifted Gr ü nwald estimates, applied to the fractional diffusion Eq. (1) and defined by (7) is unconditionally stable for $1<\alpha<2$.

Proof: We will first show that the (complex-valued) eigenvalues of the matrix A have negative real parts. Note that $\omega_1=-\alpha$ and for $1<\alpha<2$ and $t\neq 1$ we have $\omega_t>0$. Additionally,

$$-\omega_1=\alpha\ge\sum_{k=0,k\neq 1}^{k=N}\omega_k$$

for any N>1. According to the Greschgorin theorem (1), the eigenvalues of the matrix A are in the disks centered at each diagonal entry

$$A_{i,i}=\gamma_i\omega_1=-\gamma_i\alpha$$

with radius

$$r_i=\sum_{j=0,j\neq i}^{N}|A_{i,j}|=\gamma_i\sum_{j=0,j\neq i}^{i+1}\omega_{i-j+1}<\gamma_i\alpha$$

These Greschgorin disks are within the left half of the complex plane. Therefore, the eigenvalues of the matrix A have negative real-parts.

Next, λ is an eigenvalue of matrix A if and only if $(1-\lambda)$ is an eigenvalue of the matrix (I-A). If and only if $(2+\lambda)/(1-\lambda)$ is an eigenvalue of the matrix $(1-A)^{-1}(2I+A)$. We observe that the first part of this statement implies that all the eigenvalues of the matrix (I-A) have a magnitude larger than 1 and thus this matrix is invertible.

Furthermore, since the real part of λ is negative, it is not hard to check $\left|\frac{1+\lambda}{2-\lambda}\right|<1$. Therefore, the spectral radius of the system matrix $(1-A)^{-1}(2I+A)$ is less than one. Thus, the system of finite difference Eq. (7) is unconditionally stable.

NUMERICAL RESULTS

In this section the finite difference schemes (7) is used to solve the fractional wave equation (1) and with $\alpha = 1.8$.

Example: Consider the fractional wave equation of the form:

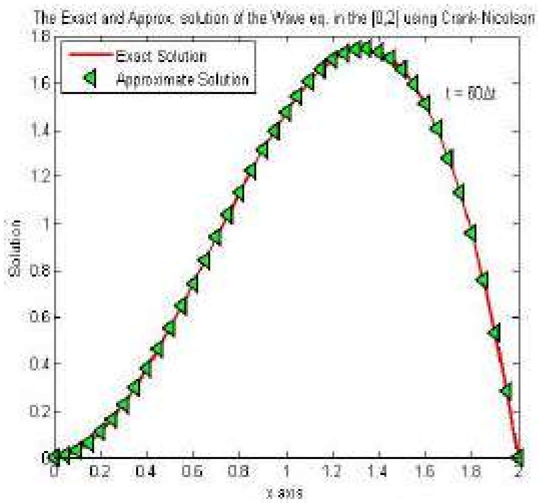
$$\frac{\partial^2u(x,t)}{\partial t^2}=c(x,t)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}}+d(x,t)$$

defined on a finite domain 0<x<2 and t>0, with the coefficient functions:

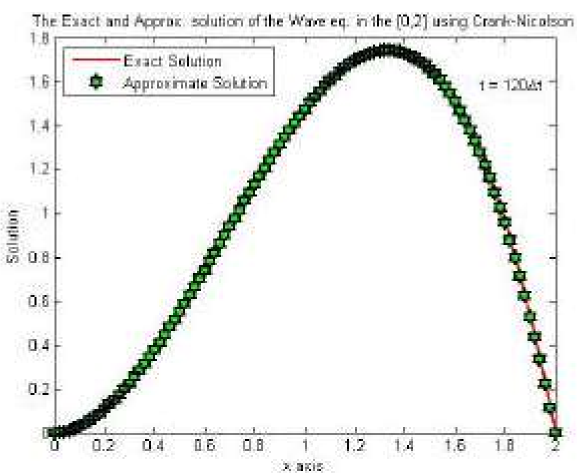
$$c(x,t)=\Gamma(1.2)x^{1.8}$$

and the source function:

$$d(x,t)=4e^{-t}x^2(2-x)-16e^{-t}x^2+20e^{-t}x^3$$



Numerical solution u (x,t) obtained at Δx = 1/40



Numerical solution u (x,t) obtained at Δx = 1/100

with initial conditions:

$$u(x,0) = 4x^2(2-x), \quad u_t(x,0) = -4x^2(2-x)$$

and Dirichlet conditions:

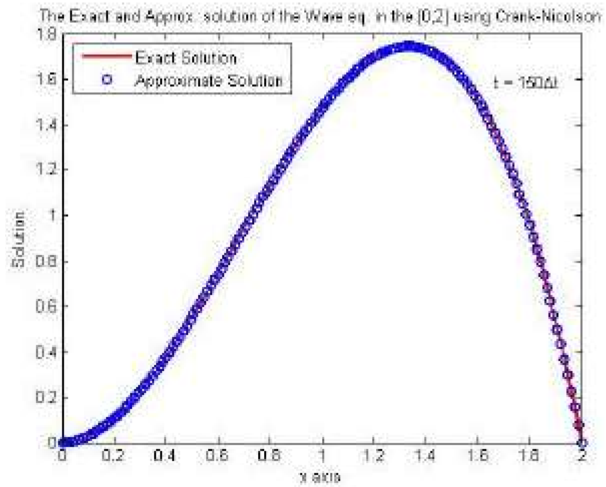
$$u(0,t) = u(2,t) = 0$$

Note that the exact solution to this problem is:

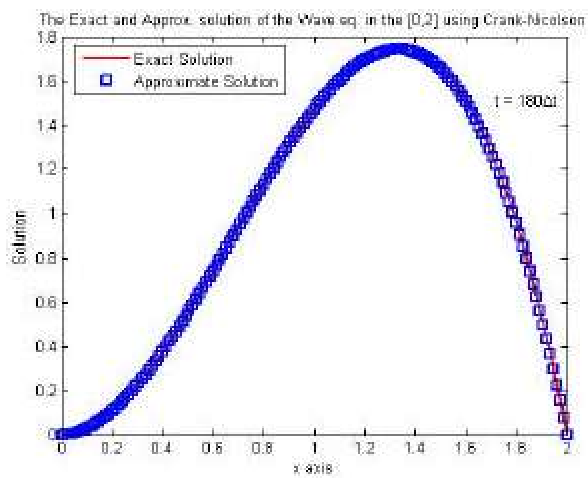
$$u(x,t) = 4e^{-t}x^2(2-x)$$

which can be verified by applying the fractional differential formula:

$$D_x^{\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}x^{m-\alpha}$$



Numerical solution u (x,t) obtained at Δx = 1/150



Numerical solution u (x,t) obtained at Δx = 1/150

Table 1: Relative error at different values Δt and Δx

Δt	Δx	Relative error
0.0250	0.1000	0.0354
0.0167	0.0500	0.0177
0.0125	0.0333	0.0122
0.0100	0.0250	0.0095
0.0083	0.0200	0.0078
0.0067	0.0133	0.0060

The obtained numerical results by means of implicit method are shown in Table 1. In the third column the relative

$$\text{error} = \frac{\left\| u^{exact} - u^{approx} \right\|_2}{\left\| u^{exact} \right\|_2}$$

is computed at time t = 1 at different values of Δt and Δx.

CONCLUSION

In this paper, a Crank-Nicholson method with shifted Grünwald estimates for solving the fractional order wave equation is presented. The stability and the consistent of the method are proved. Some test examples are given and the results obtained by the method are compared to the exact solutions. The comparison certifies that FDM gives good results. Summarizing these results, we can say that the finite difference method in its general form gives a reasonable calculations, easy to use and can be applied for the fractional differential equations in general form. All results obtained by using MATLAB version 7.1.

REFERENCES

1. Bagley, R.L. and P.J. Torvik, 1984. On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech., 51: 294-298.
2. Enelund, M. and B.L. Josefson, 1997. Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations. AIAA J., 35 (10): 1630-1637.
3. Diethelm, K., 1997. An algorithm for the numerical solution of differential equations of fractional order. Electron. Trans. Numer. Anal., 5: 1-6.
4. He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Computer Methods in Applied Mechanics and Engineering, 167 (1-2): 57-68.
5. Isaacson, E. and H.B. Keller, 1966. Analysis of Numerical Methods, Wiley, New York.
6. Jafari, H. and V. Daftardar-Gejji, 2006. Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition. Appl. Math. and Comput., 180: 488-497.
7. Lubich, Ch., 1986. Discretized fractional calculus. SIAM J. Math. Anal., 17: 704-719.
8. Meerschaert, M.M. and C. Tadjeran, 2004. Finite difference approximations for fractional advection-dispersion flow equations. J. Comput. Appl. Math., 172 (1): 65-77.
9. Meerschaert, M.M. and C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations. Appl. Numer. Math., 56: 80-90.
10. Podlubny, I., 1999. Fractional Differential Equations. Academic Press, New York.
11. Tadjeran, C. and M.M. Meerschaert, 2007. A second-order accurate numerical method for the two dimensional fractional diffusion equation. J. Comput. Phys., 220: 813-823.
12. Sweilam, N.H., M.M. Khader and R.F. Al-Bar, 2008. Homotopy perturbation method for linear and nonlinear system of fractional integro-differential equations. International Journal of Computational Mathematics and Numerical Simulation (IJCMNS)), 1 (1): 73-87.
13. Sweilam, N.H., M.M. Khader and R.F. Al-Bar, 2007. Numerical studies for a multi-order fractional differential equation. Physica Letters A, 371: 26-33.
14. Samko, S., A. Kilbas and O. Marichev, 1993. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, London.
15. Yuste, S.B., 2006. Weighted average finite difference methods for fractional diffusion equations. Journal of Computational Physics, 216: 264-274.